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Linearization of analytic isochronous centers from a given commutator ☆

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Abstract

In this paper we propose an algorithmic way to get the change of variables that linearizes an analytic isochronous center from a given commutator. Moreover, we use this method in order to obtain the linearization of some isochronous centers of the existent literature.

Keywords: Planar differential equations; Isochronous centers; Lie symmetries

1. Introduction

We consider two-dimensional analytic differential systems defined in a neighborhood $U \subset \mathbb{R}^2$ of an isolated singular point of non-degenerate *center* type, i.e. an isolated singular point with a punctured neighborhood filled of periodic orbits and with associated eigenvalues different from zero. We can do a translation of coordinates such that the critical point is located at the origin. Finally, making a linear change of coordinates, it is well known that the system can be written in the form

$$\dot{x} = -y + f(x, y), \qquad \dot{y} = x + g(x, y), \tag{1}$$

with f and g analytic functions in U starting in at least second order terms, i.e. such that $f(0,0) = g(0,0) = 0$ and $\partial_x f(0,0) = \partial_y f(0,0) = \partial_x g(0,0) = \partial_y g(0,0) = 0$. The vector field associated to system (1) will be denoted by $\mathcal{X} = (-y + f(x,y))\partial_x + (x + g(x,y))\partial_y$.

We are mainly concerned about the *isochronicity problem*, i.e., to determine whether the periodic orbits around the center have the same period (in this case the center is called *isochronous center*) or not. The main methods used in order to study isochronous center are classified in two categories. The first one, due to Sabatini and Villarini in [12] and [13] respectively, says that a center of an analytic system is isochronous if and only if there exists a commuting analytic vector field of the form $\mathcal{Y} = (x + o(x,y))\partial_x + (y + o(x,y))\partial_y$. Here commuting means $[\mathcal{X}, \mathcal{Y}] \equiv 0$ where

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the bracket used is the Lie bracket. The second one says that a center of an analytic system is isochronous if and only if there exists an analytic near-identity change of variables $(u, v) = \phi(x, y) = (x + o(x, y)), y + o(x, y)$ that linearizes $\mathcal{X}$, i.e. such that $\phi_*\mathcal{X} = -v\partial_u + u\partial v$ where ϕ_* and ϕ^* are the push-forward and pull-back defined by the analytic diffeomorphism ϕ. This last approach has been mainly used in [9,10]. We emphasize that, even for concrete isochronous systems, it is not always an easy task to obtain an explicit commutator or an explicit linearization change. Although, looking for commutators, an exception is given in [3] for f and g polynomials and system (1) having either a polynomial or a rational first integral. See [7] for the equivalence between linearizability and commutation in the more general setting of smooth or analytic vector fields in a neighborhood of non-degenerate singular points in $\mathbb{C}^n$.

Before stating the main result of this paper we want to comment that, from a linearizing change of coordinates ϕ, it is easy to get a commutator $\mathcal{Y}$ of $\mathcal{X}$ (it is just the vector field obtained by applying the inverse change of variables ϕ^{-1} to the radial field $\bar{\mathcal{Y}} = u\partial_u + v\partial_v$, that is, $\mathcal{Y} = \phi^*\bar{\mathcal{Y}}$). This is because the Lie bracket is free of coordinates, i.e., $\phi_*[\mathcal{X}, \mathcal{Y}] = [\phi_*\mathcal{X}, \phi_*\mathcal{Y}]$. As far as we know and as the authors in [4] comment, the inverse process (that is, to obtain the linearizing change of coordinates ϕ from a given commutator $\mathcal{Y}$) is an open problem. In this work we study this inverse process and we present an algorithmic method to obtain the linearization of analytic isochronous centers from a given commutator, see Theorem 4. We conclude the work showing the linearization of some extracted examples of the existent literature.

2. Lie symmetries for non-degenerate analytic centers

The following result, proved in [1] using the machinery of truncated normal forms, characterizes the centers of (1) in terms of Lie symmetries.

Theorem 1. *System* (1) *has a center at the origin if and only if there exists an infinitesimal generator of a Lie symmetry* $\mathcal{Y} = (x + \cdots)\partial_x + (y + \cdots)\partial_y$ *smooth in a neighborhood U of the origin such that* $[\mathcal{X}, \mathcal{Y}] = \mu(x, y)\mathcal{X}$*, where $\mathcal{X}$ is the associated vector field to system* (1) *and μ is a smooth scalar function in U such that* $\mu(0, 0) = 0$.

The set $N(\mathcal{X})$ of *normalizers* of $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$ is defined as the set of all the infinitesimal generators $\mathcal{Y} = \xi(x, y)\partial_x + \eta(x, y)\partial_y$ of the Lie group of symmetries of $\mathcal{X}$. In other words, $N(\mathcal{X}) = \{\mathcal{Y}\colon [\mathcal{X}, \mathcal{Y}] = \mu\mathcal{X}\}$ for some scalar function $\mu(x, y)$. The structure of $N(\mathcal{X})$ is well known (see for instance [6]): if $\bar{\mathcal{Y}} \in N(\mathcal{X})$, it can be written as $\bar{\mathcal{Y}} = H\mathcal{Y} + g\mathcal{X}$ where H is a first integral of $\mathcal{X}$ or a non-zero constant and g is any C^1 function. Moreover $[\mathcal{X}, \bar{\mathcal{Y}}] = \bar{\mu}\mathcal{X}$, with $\bar{\mu} = H\mu + \mathcal{X}g$.

For the isochronous case, there is a restricted version of Theorem 1 proved for the first time in [12].

Theorem 2. *System* (1) *has an isochronous center at the origin if and only if there exists a vector field* $\mathcal{Y} = (x + \cdots)\partial_x + (y + \cdots)\partial_y$ *analytic in a neighborhood U of the origin such that* $[\mathcal{X}, \mathcal{Y}] = 0$*, where $\mathcal{X}$ is the associated vector field to system* (1).

3. Linearizing changes of coordinates from commutators

The key point of almost the rest of the paper is the following well known fact which goes back to Poincaré and Liapunov.

Theorem 3. *System* (1) *has a center at the origin if and only if there exists a near-identity analytic change of coordinates*

$$(u, v) = \Phi(x, y) = \bigl(x + o\bigl(\bigl|(x, y)\bigr|\bigr), y + o\bigl(\bigl|(x, y)\bigr|\bigr)\bigr), \tag{2}$$

transforming system (1) *into the normal form*

$$\dot{u} = -v\bigl[1 + \Psi\bigl(u^2 + v^2\bigr)\bigr], \qquad \dot{v} = u\bigl[1 + \Psi\bigl(u^2 + v^2\bigr)\bigr], \tag{3}$$

with Ψ analytic function near the origin such that $\Psi(0) = 0$.

In the particular case that $\Psi(u^2+v^2) \equiv 0$ then the origin of system (1) is said to be an *isochronous* center because all the orbits in the period annulus have the same period, coinciding with the period 2π of the harmonic oscillator $\dot{u} = -v$, $\dot{v} = u$. See for instance [10].

We say that the $C^1(U)$ function $V(x,y)$ is an *inverse integrating factor* of a given $C^1(U)$ vector field $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y$ if the rescaled vector field $\mathcal{X}/V$ is a hamiltonian one in $U \backslash \{V^{-1}(0)\}$, that is $\mathcal{X}/V = H_y\partial_x - H_x\partial_y$. Therefore, a first integral H of $\mathcal{X}$ can be computed from the well-defined line integral

$$H(x,y) = \int \frac{Q(x,y)\,dx - P(x,y)\,dy}{V(x,y)}.$$

Remark. It is well known that, see for instance [11], if $\mathcal{Y} = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ is a normalizer of $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y$, i.e., $[\mathcal{X},\mathcal{Y}] = \mu\mathcal{X}$, then the wedge product $\mathcal{X} \wedge \mathcal{Y} := P\eta - Q\xi$ is an inverse integrating factor of $\mathcal{X}$. So, in the particular case of commutation ($\mu \equiv 0$), we get that $\mathcal{X} \wedge \mathcal{Y}$ is an inverse integrating factor of both $\mathcal{X}$ and $\mathcal{Y}$. This will be a key point in what follows.

Our main result is the following one.

Theorem 4. *Let $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y = (-y+\cdots)\partial_x + (x+\cdots)\partial_y$ and $\mathcal{Y} = \xi(x,y)\partial_x + \eta(x,y)\partial_y = (x+\cdots)\partial_x + (y+\cdots)\partial_y$ be two analytic vector fields in a neighborhood U of the origin such that $[\mathcal{X},\mathcal{Y}] = 0$. Then, a near-identity change of variables $u = x+\cdots$, $v = y+\cdots$, analytic in U that linearizes $\mathcal{X}$ is obtained as follows*:

$$u = \frac{\sqrt{f(H)}\,g(I)}{\sqrt{1+g^2(I)}}, \qquad v = \frac{\sqrt{f(H)}}{\sqrt{1+g^2(I)}}, \tag{4}$$

where H and I are first integrals of $\mathcal{X}$ and $\mathcal{Y}$, respectively, associated with the inverse integrating factor $\mathcal{X} \wedge \mathcal{Y}$ and f and g are two functions such that $f(H(x,y)) = x^2+y^2+\cdots$ and $g(I(x,y)) = (y+\cdots)/(x+\cdots)$.

Proof. The existence of the linearizing change of variables $(x,y) \to (u,v)$ is known. The new result is how to obtain the linearizing change of variables from the knowledge of the commutator $\mathcal{Y}$.

From (4) we have that $f(H) = u^2+v^2$ and $g(I) = v/u$ are first integrals of $\mathcal{X}$ and $\mathcal{Y}$ respectively, i.e. $\mathcal{X}f(H) = \mathcal{Y}g(I) \equiv 0$. It follows

$$u\mathcal{X}(u) + v\mathcal{X}(v) \equiv 0, \qquad u\mathcal{Y}(v) - v\mathcal{Y}(u) \equiv 0.$$

Therefore we have

$$\mathcal{X}(u) = -v\Lambda, \qquad \mathcal{X}(v) = u\Lambda, \qquad \mathcal{Y}(u) = u\Omega, \qquad \mathcal{Y}(v) = v\Omega, \tag{5}$$

with $\Lambda(x,y)$ and $\Omega(x,y)$ analytic functions in a neighborhood of the origin. From the above equations we observe that, if we prove that Λ is a non-vanishing constant then the change $(x,y) = (u(x,y), v(x,y))$ linearizes the vector field $\mathcal{X}$.

On the other hand, since $[\mathcal{X},\mathcal{Y}] = 0$, in particular we have

$$\mathcal{X}\mathcal{Y}(u) - \mathcal{Y}\mathcal{X}(u) \equiv 0, \qquad \mathcal{X}\mathcal{Y}(v) - \mathcal{Y}\mathcal{X}(v) \equiv 0.$$

Introducing (5) in the former relations we get

$$u\mathcal{X}(\Omega) + v\mathcal{Y}(\Lambda) = 0, \qquad v\mathcal{X}(\Omega) - u\mathcal{Y}(\Lambda) = 0.$$

This is a linear homogeneous algebraic system for the unknowns $\mathcal{X}(\Omega)$ and $\mathcal{Y}(\Lambda)$ with associated determinant $-(u^2+v^2)$ which is different from zero out of the origin. So the unique solution is the trivial one $\mathcal{X}(\Omega) = \mathcal{Y}(\Lambda) \equiv 0$. From this last equality, Λ is either a constant or a first integral of $\mathcal{Y}$ in a neighborhood of the origin. But, this point is a singular point of $\mathcal{Y}$ of type node and so there are no continuous first integral of $\mathcal{Y}$ in that neighborhood in contradiction with the fact that Λ is analytic. Hence the only possibility is Λ equal to a constant different from zero because otherwise, from (5), u and v would be analytic first integrals of $\mathcal{X}$ which is impossible. $\square$

Example 1. Quadratic isochronous centers were classified by Loud [8]. Hence, it is known that the quadratic system

$$\dot{x} = -y - \frac{4}{3}x^2, \qquad \dot{y} = x\left(1 - \frac{16}{3}y\right), \tag{6}$$

has an isochronous center at the origin. Moreover, it is shown (see p. 34 of [2]) that the associated vector field $\mathcal{X}$ to system (6) commutes with $\mathcal{Y}_1 = 3x(9 - 24y + 32x^2)\partial_x + (3y + 4x^2)(9 - 24y + 32x^2)\partial_y$. Taking $\mathcal{Y} = \mathcal{Y}_1/27$ we have $\mathcal{Y} = (x + \cdots)\partial_x + (y + \cdots)\partial_y$ and $[\mathcal{X}, \mathcal{Y}] = 0$. Then $V = \mathcal{X} \wedge \mathcal{Y} = (9 + 32x^2 - 24y)(9x^2 + 16x^4 - 24x^2y + 9y^2)$ up to multiplicative constants and $H(x, y)$ and $I(x, y)$ can be computed (and simplified after deleting some arctan and log functions) to

$$H(x, y) = \frac{9x^2 + 16x^4 - 24x^2y + 9y^2}{(9 + 32x^2 - 24y)^2}, \qquad I(x, y) = \frac{-3y + 4x^2}{3x}.$$

Since $H(x, y) = (x^2 + y^2)/9 + \cdots$ we take $f(H) = 9H$. Moreover $g(I) = -I$ so that

$$g\big(I(x, y)\big) = \frac{y - \frac{4}{3}x^2}{x}.$$

Finally, from (4), we get the linearizing change of coordinates

$$u = \frac{9x}{9 + 32x^2 - 24y}, \qquad v = \frac{3(3y - 4x^2)}{9 + 32x^2 - 24y},$$

according to [2]. We note that this change of variables linearizes both vector fields $\mathcal{X}$ and $\mathcal{Y}$.

The authors of the survey paper [2] have carried out a quite exhaustive classification of several families of isochronous systems. Also, they give the linearizing changes and commutators in many cases. In any way, some incomplete examples exist in their tables. In the following two examples we obtain the linearizing change that lacks in some of these systems by using our Theorem 4.

Example 2. In Table 10 of [2] the authors present the cubic reversible system

$$\dot{x} = -y(1 - x)(1 - 2x), \qquad \dot{y} = x - 2x^2 + y^2 + 2x^3, \tag{7}$$

having an isochronous center at the origin. Moreover, they prove that the associated vector field $\mathcal{X}$ to system (7) commutes with $\mathcal{Y} = (1 - x)(x - 2x^2 + 2y^2 + 2x^3 - 2xy^2)/(1 - 2x)\partial_x + y(1 - x)(1 - 4x + 6x^2 + 2y^2)/(1 - 2x)\partial_y = (x + \cdots)\partial_x + (y + \cdots)\partial_y$. Then $V = \mathcal{X} \wedge \mathcal{Y} = (1 - x)(x^2 + y^2)(1 - 4x + 8x^2 - 8x^3 + 4x^4 + 4y^2 - 8xy^2 + 4x^2y^2)/(-1 + 2x)$ is an inverse integrating factor for both vector fields $\mathcal{X}$ and $\mathcal{Y}$. The first integrals $H(x, y)$ and $I(x, y)$ associated with V of $\mathcal{X}$ and $\mathcal{Y}$ respectively are then calculated (and simplified after deleting arctan and arctanh functions). We get

$$H(x, y) = \frac{1 - 4x + 12x^2 - 16x^3 + 8x^4 + 8(-1 + x)^2y^2}{(1 - 2x)^2},$$
$$I(x, y) = \frac{y}{x - 2x^2 + 2x^3 - 2y^2 + 2xy^2}.$$

Since $H(x, y) = 1 + 8(x^2 + y^2) + \cdots$ we take $f(H) = (H - 1)/8 = x^2 + y^2 + \cdots$. Moreover $g(I) = I$ so that, taking into account (4), we get the linearizing change of coordinates

$$u = \big(x + 2\big(-x^2 + x^3 - y^2 + xy^2\big)\big)\Delta(x, y), \qquad v = y\Delta(x, y),$$

where $\Delta(x, y) := (x - 1)/[(1 - 2x)\sqrt{(1 + 2x(x - 1))^2 + 4(x - 1)^2y^2}]$. We note that this change of variables only linearizes $\mathcal{X}$.

Example 3. The system labeled as $H4_4$ in Table 13 of [2] is the following linear center perturbed by a homogeneous polynomial of fourth degree

$$\dot{x} = -y - \frac{4}{9}x^4 - \frac{20}{9}x^2y^2, \qquad \dot{y} = x + \frac{40}{9}x^3y + \frac{16}{9}xy^3. \tag{8}$$

The origin is an isochronous center and in [2] is proved that its associated vector field $\mathcal{X}$ commutes with $\mathcal{Y}_1 = f_1(x,y)[x(3+8x^2y)\partial_x + (3y-12x^4-4x^2y^2)\partial_y]$ where $f_1(x,y) = 9+24y(x^2+y^2)+32x^2(x^2+y^2)^2$. We define $\mathcal{Y} = \mathcal{Y}_1/27$ so that $\mathcal{Y} = (x+\cdots)\partial_x + (y+\cdots)\partial_y$ and $[\mathcal{X},\mathcal{Y}] = 0$. Then $V = \mathcal{X} \wedge \mathcal{Y} = (x^2+y^2)f_1(x,y)f_2(x,y)$ up to constants is an inverse integrating factor for both vector fields $\mathcal{X}$ and $\mathcal{Y}$. Here $f_2(x,y) = 9+16x^6+24x^2y+16x^4y^2$. The first integral of $\mathcal{X}$ associated with V is $H(x,y) = (x^2+y^2)^3 f_1^{-2} f_2$. Hence a first integral good for our purpose is just $f(H) = 3^{2/3}H^{1/3} = x^2+y^2+\cdots$. In addition, $\mathcal{Y}$ possesses, associated to V, the first integral $I(x,y) = (3y+4x^4+4x^2y^2)/(3x) = (y+\cdots)/(x+\cdots)$ (after simplifying an arctan function). In short, $g(I) = I$, and due to (4), the change of coordinates

$$u = 3^{4/3}x(f_1f_2)^{-1/3}, \qquad v = 3^{1/3}\big[3y+4x^2\big(x^2+y^2\big)\big](f_1f_2)^{-1/3},$$

linearizes both $\mathcal{X}$ and $\mathcal{Y}$.

The last example comes from the work [5]. In that paper it is proved that the potential vector field $\mathcal{X} = -y\partial_x + V'(x)\partial_y$ has an isochronous center at the origin for the rational potential function

$$V(x) = \frac{x^2(x-2)^2}{(x-1)^2}, \tag{9}$$

and they comment that it is difficult to obtain an explicit commutator and a linearizing change of coordinates. We will obtain in the next example both characterizations of isochronicity for such a system.

Example 4. Let us consider the potential systems

$$\mathcal{X} = -y\partial_x + V'(x)\partial_y,$$

with hamiltonian function of type kinetic plus potential $H = y^2/2 + V(x)$ and $V(x) = x^2/2 + o(x^2)$ analytic function having a minimum at the origin. Hence, the origin is a center of $\mathcal{X}$. One can check by straightforward calculations (this formula appears in [5]) that $[\mathcal{X},\mathcal{Y}] = \mu\mathcal{X}$, where

$$\mathcal{Y} = \frac{V(x)}{V'(x)}\partial_x + \frac{y}{2}\partial_y, \qquad \mu(x) = \frac{(V'(x))^2 - 2V(x)V''(x)}{2(V'(x))^2}.$$

First of all we note that if $g(x,y)$ is a C^1 solution of the partial differential equation $\mathcal{X}g = -\mu H$, then $\mathcal{Y}^* = H\mathcal{Y} + g\mathcal{X}$ satisfies $[\mathcal{X},\mathcal{Y}^*] = 0$. Assume that $V(x)$ is given by (9) one obtains

$$g(x,y) = \frac{(1-x)yA(x,y)}{2(2-2x+x^2)B(x,y)},$$

where $A(x,y) = 8x^2+y^2-8x^3+2x^4-2xy^2+x^2y^2$ and $B(x,y) = 8-16x+16x^2+y^2-8x^3+2x^4-2xy^2+x^2y^2$. Therefore

$$\mathcal{Y}^* = \frac{A(x,y)C(x,y)}{4(x-1)B(x,y)}\partial_x + \frac{yA(x,y)(8+C(x,y))}{4(x-1)^2B(x,y)}\partial_y$$

where $C(x,y) = -8x+12x^2+y^2-8x^3+2x^4-2xy^2+x^2y^2$. Since $\mathcal{Y}^* = (xy^2/4+2x^3+\cdots)\partial_x + (y^3/4+2x^2y+\cdots)\partial_y$ we take

$$\bar{\mathcal{Y}} = \frac{\mathcal{Y}^*}{H} = \frac{(x-1)C(x,y)}{2B(x,y)}\partial_x + \frac{y(8+C(x,y))}{2B(x,y)}\partial_y = \left(\frac{x}{2}+\cdots\right)\partial_x + \left(\frac{y}{2}+\cdots\right)\partial_y,$$

which also satisfies $[\mathcal{X},\bar{\mathcal{Y}}] = 0$ and has a star node at the origin.

Now taking into account that $\mathcal{X} \wedge \bar{\mathcal{Y}}$ is an inverse integrating factor of $\bar{\mathcal{Y}}$ it is easy to obtain for $\bar{\mathcal{Y}}$ the next first integral

$$I(x,y) = \frac{y(1-x)^3}{D(x,y)} = \frac{y+\cdots}{8x+\cdots},$$

where $D(x, y) = 8x - 12x^2 + y^2 + 8x^3 - 2x^4 - 2xy^2 + x^2y^2$. Finally, taking $f(H) = 2H = 8x^2 + y^2 + \cdots$, and solving the system $8u^2 + v^2 = f(H)$, $v/(8u) = I$ we obtain that the change of coordinates

$$u(x, y) = \frac{-D(x, y)}{2\sqrt{2}(x-1)\sqrt{B(x, y)}} = x + \cdots, \qquad v(x, y) = \frac{2\sqrt{2}y(x-1)^2}{\sqrt{B(x, y)}} = y + \cdots$$

brings $\mathcal{X}$ to the linear vector field $-v\partial_u + 8u\partial_v$.

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